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THE EVOLUTION OF A PLASTIC ZONE NEAR A HOLE†

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An elastoplastic problem (EPP) is considered in the case when the load on part of the boundary of the domain does not vary. A plastic zone is adjacent to this part of the boundary, and the stress field in this zone is often considered to be known in advance. This problem then reduces to the problem of matching the plastic stress field with the required elastic field subject to the condition of the continuity of the stresses in the required matching contour. The elastic stress field which is determined here may, however, extend beyond the yield surface. For example, in the case of the well-known problem of the biaxial loading of a plane with a hole [1], this occurs when the load reaches a certain critical value but not a limit one [2]. In the case of a superficial load, the solution of the matching problem is not a solution of an EPP. In this case, a new stress field has to be constructed in a part of the plastic zone in order to solve the EPP.

In the case of a supercritical load, which only differs slightly from a critical load, this problem is solved by the small-parameter method. Here, the shape of the plastic zone is also determined. To be specific, we consider the problem of a plastic zone which wholly encompasses the contour of the hole to which a non-varying load is applied. A similar construction can be used for other planar EPPs with critical loads. The calculation of the plastic zone which encompasses a circular hole in a plane which is subject to biaxial loading at infinity is presented as an example.

1. THE CRITICAL LOAD IN THE MATCHING PROBLEM

SUPPOSE an elastoplastic body is under conditions of plane strain. The corresponding plane problem for a plane with a hole or for a bounded domain with a hole is considered. The load is described by forces which are imposed on the contour of the hole and at infinity (or on the outer boundary of the domain, respectively). Let the hole be completely encompassed by a plastic zone Ω^p which is separated from the elastic zone Ω^e by the contour l (see Fig. 1). The components of the stress tensor σ_x, σ_y and σ_{xy}, in an x, y Cartesian system of coordinates satisfy the equilibrium equations and the yield criterion

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}=0,\quad \frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}=0,\quad F(\sigma)\equiv(\sigma_y-\sigma_x)^2+4\,\sigma_{xy}^2-1=0 \tag{1.1}$$

in the plastic zone. The stresses are assumed to be dimensionless. They are reduced with respect to the quantity $2k$, where k is the yield limit under pure shear.

The load on the contour of the hole subsequently remains constant.

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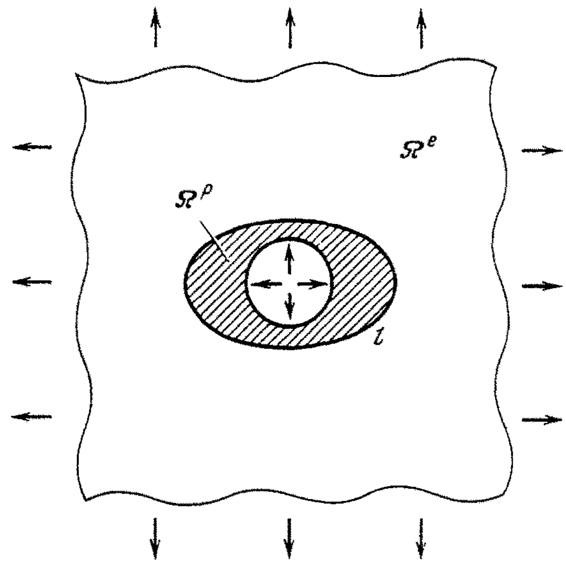

FIG. 1.

It is usually assumed that Eqs (1.1) together with the force boundary conditions on the contour of the hole completely define the stresses in the plastic zone. This stress field is denoted by σ^p and is assumed to be known.

In the elastic zone, the stresses are defined by a biharmonic stress function. The condition of continuity of the stresses enables us to express the second derivatives of this function on the unknown matching contour l in terms of the known stresses σ^p in the plastic zone.

The problem of finding the matching contour l and the biharmonic stress function which satisfies these equations and the specified loading conditions at infinity (or on the outer boundary of the domain) will henceforth be referred to as the matching problem. Starting out from [1], where an analytic solution of the matching problem was constructed in the case of a plane with a circular hole, analytic and numerical methods have been developed for solving it. A review of these methods can be found in [3, 4].

The solution of the matching problem is not always the solution of the elastoplastic problem.

Example. Suppose a plane with a load-free circular hole is subjected at infinity to the action of a force p along the x axis and a force q along the y axis (like the stresses, the forces are also made dimensionless)

$$\sigma_x \to p,\ \sigma_y \to q,\ \sigma_{xy} \to 0 \quad \text{as} \quad x^2+y^2 \to \infty$$

The solution of the matching problem and, in particular, the stress field σ^p, is constructed [1] subject to the condition

$$|q-p| < 1,\quad (1-|q-p|)\exp\{\tfrac{1}{2}(|p+q|-1)\} \geqslant 1 \tag{1.2}$$

However, it can only be used as the solution of the elastoplastic problem when the auxiliary condition $|q-p| \leqslant \sqrt{(2)}-1$ is satisfied.

The reason for this firstly lies in the fact that, when $|q-p| > \sqrt{(2)}-1$, there are characteristics of (1.1), constructed for the stress field σ^p, which intersect the matching contour at three points. On account of this, there is no corresponding solution for the displacements [4]. The families of characteristics and the matching contour l for the solution in [1] are shown in Fig. 2 for various $q > p > 0$, which satisfy conditions (1.2), in the coordinates $\ln r/a$ and φ (r and φ are polar coordinates in the x, y plane and a is the radius of the hole). The cases when $q-p < \sqrt{(2)}-1$, $q-p = \sqrt{(2)}-1$, and $q-p > \sqrt{(2)}-1$ are shown in Fig. 2(a–c), respectively.

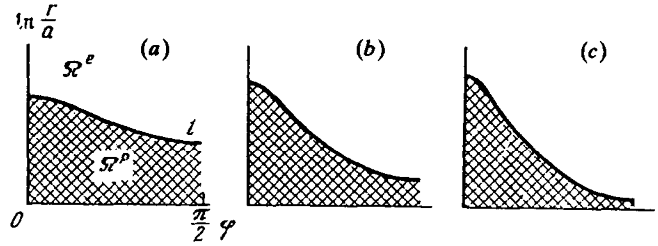

FIG. 2.

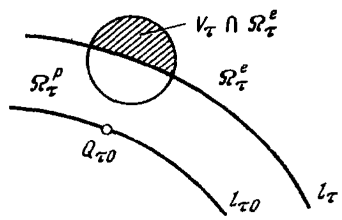

FIG. 3.

Secondly, when $|q-p|>\sqrt{(2)}-1$, the stress field constructed in [1] extends in a part of the domain Ω^e, beyond the yield surface [2]. At values of $|q-p|$ close to $\sqrt{(2)}-1$, this occurs in the neighbourhood of the point of contact of a characteristic and the contour l corresponding to a value of $|q-p|=\sqrt{(2)}-1$. This link between the stresses extending beyond the yield surface and the contact of a characteristic with the matching contour also holds in the case considered in Sec. 2.

Let us now consider a matching problem when the load at infinity (or on the outer surface of a domain) is governed by a monotonically increasing dimensionless parameter τ, while the load on the contour of the hole is constant. By virtue of the latter condition, the stress field σ^p in the zone Ω^e is independent of the parameter τ, The zones Ω^p, Ω^e and the matching contour l depend on the parameter τ and are henceforth denoted by Ω_τ^p, Ω_τ^e and l_τ.

When $\tau \leqslant \tau_0$, let the condition of the admissibility of the stresses $F(\sigma)\leqslant 0$ be satisfied in the case of the solution of the matching problem in the zone Ω_τ^e and, for τ close to τ_0, $\tau>\tau_0$, let there be a point Q_τ on the contour l_τ such that the opposite inequality $F(\sigma)>0$ holds at the intersection of a certain neighbourhood V_τ of it with the zone Ω_τ^e; the stresses extend beyond the yield surface (see Fig. 3). We shall call the value of τ_0 and the load corresponding to it the critical values, while the loads which correspond to $\tau<\tau_0$ ($\tau>\tau_0$) are referred to as subcritical (supercritical). In the example being considered, for the load (τp_0, τq_0) satisfying conditions (1.2) the value $\tau_0=(\sqrt{(2)}-1)/|q_0-p_0|$ is critical.

In the case of supercritical loads, the solution of the matching problem does not yield a stress field which is a solution of the elastoplastic problem. The aim of this paper is to construct a stress field which is a solution of the elastoplastic problem in the case of a supercritical load when $\tau-\tau_0 \ll 1$.

2. THE CRITICAL LOAD AND THE CONTACT OF A CHARACTERISTIC WITH THE MATCHING CONTOUR

We will point out a property of the critical load which is important in the ensuing discussion (it has been mentioned in the example in Sec. 1 and also holds in the general case). It is namely that, at the critical load, contact occurs between the characteristics of system (1.1) and the matching contour l_{τ_0}.

In order to prove this property, let us first consider F_τ, the function in (1.1), for the case of the stress field σ_τ which is the solution of the matching problem in the zone Ω_* at a value τ of the loading parameter. Let $\partial F_\tau/\partial n$ be its derivative along the direction of the normal to the contour l_τ which is external with respect to the domain Ω_τ^p. Let τ_0 be the critical value of the loading parameter, Q_τ, where $\tau>\tau_0$ is the point indicated in the definition of the critical load, and let Q_{τ_0} be the point on the contour l_{τ_0} which is the limit point for points Q_τ when $\tau\to\tau_0$.

We will show that the derivative

$$\partial F_{\tau_0}/\partial n|_{Q_{\tau_0}}=0 \tag{2.1}$$

vanishes at this point.

Actually, the equality $F_\tau=0$ always holds on the contour l_τ by virtue of the continuity of the stresses. Hence, in the case when $\tau<\tau_0$, when $F_\tau\leqslant 0$ in the zone Ω_τ^e, the inequality $\partial F_\tau/\partial n\leqslant 0$ is satisfied at all points of the contour l_τ. Likewise, in the case when $\tau>\tau_0$, when $F_\tau>0$ in the domain $V_\tau\cap\Omega_\tau^e$, the inequality $\partial F_\tau/\partial n\geqslant 0$ is satisfied at the point Q_τ. When $\tau=\tau_0$, Eq. (2.1) holds at the point Q_{τ_0} by continuity.

Allowing for the fact that the relationship $F_\tau=0$ is satisfied in the zone Ω_τ^p, we represent the equality (2.1) in the form $[\partial F_{\tau_0}/\partial n]|_{Q_{\tau_0}}=0$ (here and subsequently, a discontinuity in a quantity is denoted by square brackets).

We will now show that contact between a characteristic and the contour l_{τ_0} at the point Q_{τ_0} follows from this equality. We select a system of Cartesian coordinates with the origin at the point Q_{τ_0} with the y axis directed along the normal and the x axis directed along the tangent to the contour l_{τ_0}. Then, according to formula (1.1), in the case of the function F_τ, the preceding relationship is represented in the form (when $x=0,\ y=0$)

$$(\sigma_y-\sigma_x)\left([\partial\sigma_y/\partial y]-[\partial\sigma_x/\partial y]\right)+4\,\sigma_{xy}\,[\partial\sigma_{xy}/\partial y]=0$$

When account is taken of the relationships (when $x=0,\ y=0$)

$$[\partial\sigma_{xy}/\partial y]=0,\quad [\partial\sigma_y/\partial y]=0$$

which follow from the equilibrium equations (1.1), it is equivalent to the equality (when $x=0,\ y=0$)

$$(\sigma_y-\sigma_x)\,[\partial\sigma_x/\partial y]=0$$

If the second factor differs from zero, the equalities $\sigma_y-\sigma_x=0$ and $\sigma_{xy}=\pm\tfrac{1}{2}$ are satisfied at the point Q_{τ_0}. This means [5] that the directions of the x and y axes are characteristics for system (1.1) at the point Q_{τ_0}.

Hence, in the case of the critical load $\tau=\tau_0$, the characteristic of system (1.1) is tangent to the matching contour l_{τ_0} at the point Q_{τ_0} only if the discontinuity $[\partial\sigma_y/\partial y]$ is non-zero at this point. The opposite assertion also holds, namely, when $\tau\leqslant\tau_*$, let the solution of the matching problem in the domain Ω_τ^e, satisfy the condition of the admissibility of the stresses $F(\sigma)\leqslant 0$ and, when $\tau=\tau_*$, the characteristic of system (1.1) is tangent to matching contour l_{τ_*} at the point Q_{τ_*}.

We will show that, in the general case, the value of τ_* is the critical value. Consider the function $f_\tau=\partial F_\tau/\partial n$ for a value of the parameter τ on the contour l_τ close to τ_*. When $\tau\leqslant\tau_*$, the inequality $f_\tau\leqslant 0$ is satisfied at all points of the contour l_τ and, in particular $f_{\tau_*}\leqslant 0$. At the point Q_{τ_*}, the characteristic is tangent to the contour l_{τ_*} and this means that the equality $(\sigma_y-\sigma_x)(Q_{\tau_*})=0$ holds in the same local Cartesian system of coordinates as was used above. The relationship $f_{\tau_*}(Q_{\tau_*})=0$ follows from it. The function F_{τ_*} therefore has a local maximum at the point Q_{τ_*}. A local maximum of the function f_τ at a value of the parameter τ close to τ_* corresponds to it. Let us define the function $g(\tau)=\max f_\tau$. This function is positive for values of $\tau<\tau_*$ and vanishes when $\tau=\tau_*$. Hence, in the general case (when $(dg/d\tau)(\tau_*)\neq 0$), it is positive when $\tau>\tau_*$. The inequality $(\partial F_\tau/\partial n)(Q_\tau)>0$ is then satisfied at the point Q_τ at which the function f_τ reaches a maximum. The inequality $F_\tau>0$ is therefore satisfied in the domain Ω_τ^e close to the point Q_τ. Consequently, the critical load corresponds to the value of τ at which the characteristic and the matching contour are tangent to one another.

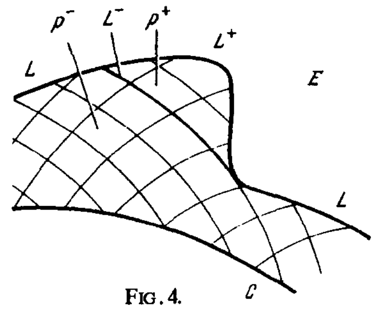

FIG. 4.

Then, the characteristics are not tangent to the matching contour in the case of a subcritical load but intersect it at a single point. Hence, the contact which occurs at the critical load between the characteristic and the matching contour is of the second order (see Fig. 2).

Remark. The properties which have been established refer to the general case which is also considered later. It is possible, however, that they will break down when there is a certain degeneracy. This occurs when $(dg/d\tau)(\tau_*)=0$. Then, when $\tau>\tau_*$, the function $g(\tau)$ may take not positive values, as was considered above, but negative values. It is therefore necessary to check that there is no degeneracy when applying the properties which have been established to the analysis of an actual problem.

3. THE PLASTIC ZONE OF THE SOLUTION OF AN ELASTOPLASTIC PROBLEM IN THE CASE OF A SUPERCRITICAL LOAD

The plane elastoplastic problem has a unique solution [6] in the case of a supercritical load if it is below the critical load. The stress field of this solution in the corresponding plastic zone is not identical with the field σ^p everywhere. In fact, if this were to be so, the solution of the elastoplastic problem would also be the solution of the matching problem. However, the latter is also unique and, in the case of a supercritical load, passes beyond the yield surface, which means that it cannot be identical with the solution of the elastoplastic problem.

Hence, in the case of a supercritical load, the stress field σ^p is only preserved in a certain part P^- of the plastic zone of the solution of the elastoplastic problem. Each point of the domain P^- is joined to the contour of the hole by curves of both families of characteristics of system (1.1) which lie within the plastic zone. In other domains of the plastic zone, where the points cannot possibly be joined in this manner with the contour of the hole, the stress field s differs from σ^p. Let P^+ be such a domain and let L^- be the curve which separates this domain and the domain P^-.

The curve L^- is a characteristic of system (1.1). Actually, if this were not so, the stress field σ^p would propagate into the domain P^+ (no other continuous extension of this solution of system (1.1) through a non-characteristic curve is possible). This, however, contradicts the choice of the domain P^+. Hence, the domain P^+ is separated from the domain P^- by the characteristics of system (1.1). We shall henceforth confine ourselves to the case when the domain P^+ under consideration is separated by a single characteristic (Fig. 4).

4. THE SMALL-PARAMETER METHOD IN AN ELASTOPLASTIC PROBLEM IN THE CASE OF A SUPERCRITICAL LOAD

Let τ_0 be the critical value of the loading parameter. We will consider a supercritical load, which corresponds to a value of $\tau=\tau_0+\epsilon^2$, $\epsilon^2\ll 1$. It is required to find the stress field which is the solution of the elastoplastic problem in the case of this load.

Let E_ϵ and P_ϵ be the corresponding elastic and plastic zones, and P_ϵ^- and P_ϵ^+ be the parts of the plastic zone in which σ^p and the field which differs from it s_ϵ (Sec. 3) are the stress fields, respectively. We will denote the stress field in the elastic zone by σ_ϵ^e and the curves, which separate the pairs of domains P_ϵ^- and P_ϵ^+, P_ϵ^+ and E_ϵ, P_ϵ^- and E_ϵ by L_ϵ^-, L_ϵ^+ and L_ϵ, respectively (see Fig. 4 where the subscript ϵ is omitted).

For the critical load $\epsilon = 0$, the solution of the matching problem is the solution of the elastoplastic problem: the domain P_0^+ and the curves L_0^- and L_0^+ are absent, the domains P_0^- and E_0, the contour L_0 and the stress field σ_0^e are, respectively, identical with $\Omega_{\tau_0}^p$ and $\Omega_{\tau_0}^e$, the contour l_{τ_0} the stress field σ_{τ_0}. It is required to find the solution of the elastoplastic problem when $\epsilon \neq 0$, that is, to find the curves L_ϵ^-, L_ϵ^+ and L_ϵ and the field of the stresses s_ϵ and σ_ϵ^e such that (1) the system of equations (1.1) of the theory of plasticity is satisfied in the domain P_ϵ^+ when $\sigma = s_\epsilon$, (2) in the domain E_ϵ, the stress field $\sigma = \sigma_\epsilon^e$ is expressed in terms of a biharmonic stress function and satisfies the admissibility condition $F(\sigma) \leq 0$, (3) the matching condition, that is, the condition of the continuity of the stresses is satisfied on the contours L_ϵ^-, L_ϵ^+ and L_ϵ, and (4) the boundary conditions for the stresses under a load corresponding to the value $\tau = \tau_0 + \epsilon^2$ of the load parameter are satisfied at infinity (or on the outer boundary of the domain). (We recall that the stress field $\sigma = \sigma^p$ which satisfies system (1.1) and a fixed boundary condition on the contour of the hole is determined in the domain P_ϵ^-.)

A number of problems both with complete and incomplete envelopment of the hole by a plastic zone is solved by small-parameter method [7, 8]. In the case of a supercritical load some additional treatment is required. This is associated with the determination of the stresses in the zone P_ϵ^0 and with checking the admissibility of the stresses in the zone E_ϵ.

The system of coordinates. In solving the problem in question by the small-parameter method we shall subsequently use a special system of coordinates which is associated with the contour $L_0 \equiv L_{\tau_0}$. Let x be the length of an arc measured from the point Q_{ϵ_0}, $\mathbf{r}_0(x)$ by the vectorial parametric specification of this contour, and $\mathbf{n}(x)$ be the unit normal to it which is external with respect to the domain $\Omega_{\tau_0}^p$. We will introduce an orthogonal curvilinear system of coordinates x, y by correlating a point in the plane with a radius vector $\mathbf{r}(x, y) = \mathbf{r}_0(x) + \mathbf{n}(x)y$ with the pair (x, y).

We shall denote the dimensionless physical components of the stress tensor in this system of coordinates, which are referred to the quantity $2k$, by σ_x, σ_y and σ_{xy}. Here, the yield condition preserves its previous form while the equilibrium equations takes the form

$$\frac{\partial \sigma_x}{\partial x} + \left(1 + \frac{y}{R(x)}\right)\frac{\partial \sigma_{xy}}{\partial y} + \frac{2}{R(x)}\sigma_{xy} = 0 \tag{4.1}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \left(1 + \frac{y}{R(x)}\right)\frac{\partial \sigma_y}{\partial y} + \frac{\sigma_y - \sigma_x}{R(x)} = 0$$

where $R(\mathbf{x})$ is the radius of curvature of the contour L_0.

Representation of the boundaries of the plastic zone. The segment L_ϵ^- of the boundary of the domain P_ϵ^- is a characteristic (Sec. 3). Hence, if there is no finite difference between the angles of inclination of L_ϵ^- and the limiting curve for it L_0, then L_ϵ^0 is located close to the point Q_{τ_0} $(x = 0,\ y = 0)$ at which the matching contour and the characteristic are tangent to one another under the critical load and L_ϵ^- belongs to this family of characteristics.

Hence, the zone P_ϵ^+ lies close to the point Q_{τ_0}. Let us estimate the characteristic dimensions of this zone. Let $\sigma = \sigma_{\tau_0+\epsilon^2}(x, y)$ be the stress field of the solution of the matching problem. It reaches the yield surface in a certain domain $V_{\tau_0+\epsilon^2}$ close to the point Q_{τ_0} and, in the x direction, the domain has dimensions of the order of ϵ while, in the y direction, the dimensions are of the order of ϵ^2. The extent to which the stresses extend beyond the yield surface is characterized by the relationship $F(\sigma) = O(\epsilon^4)$ (Sec. 6). This means that the stress field under consideration may also be taken as a first approximation of the solution of the elastoplastic problem both in the zone E_ϵ as well as in the zone P_ϵ^+ with an error of the order of ϵ^3. The zone P_ϵ^+ therefore has the same characteristic dimensions as the domain $V_{t_0+\epsilon^2}$.

Let $y=\rho_\epsilon^-(x)$ be the equation of the curve L_ϵ^-. Since this curve is a characteristic, the form of the function $\rho_\epsilon^-(x)$ is determined by the quantity $r=r_2\epsilon^2+r_3\epsilon^3+\dots$, that is, the value of this function when $x=0$. Let $y=f(x,\eta)$ be the equation of a characteristic of the same family as L_ϵ^- which passes through the point $x=0$, $y=\eta$. In accordance with the estimate of the size of the zone P_ϵ^+, let us expand the function $f(x,\eta)$ in the neighbourhood of the point $x=0$, $y=0$ allowing for the fact that $f(0,\eta)=\eta$ and that, as a consequence of the second order contact between the characteristic and the contour L_0 (Sec. 2), the relationships

$$f(0,0)=0,\ \frac{\partial f}{\partial x}(0,0)=0,\ \frac{\partial^2 f}{\partial x^2}(0,0)=0$$

are satisfied at the point $x=0$, $\eta=0$.

Using the expansion, the equation of the curve L_ϵ^- is represented in the form

$$y=\rho_\epsilon^-(x)=\epsilon^2\rho_2^-\left(\frac{x}{\epsilon}\right)+\epsilon^3\rho_3^-\left(\frac{x}{\epsilon}\right)+\dots$$

$$\rho_2^-\left(\frac{x}{\epsilon}\right)=\text{coust}=r_2,\ \rho_3^-\left(\frac{x}{\epsilon}\right)=\frac{1}{6}f_{xxx}\frac{x^3}{\epsilon^3}+f_{x\eta}r_2\frac{x}{\epsilon}+r_3 \tag{4.2}$$

$$f_\eta=\frac{\partial f}{\partial \eta}(0,0),\ f_{x\eta}=\frac{\partial^2 f}{\partial x\partial \eta}(0,0),\ f_{xxx}=\frac{\partial^3 f}{\partial x^3}(0,0)$$

The quantities ρ_2^-, ρ_3^-, ..., in accordance with the estimate of the dimensions of the zone P_ϵ^+, are of the order of unity. The numbers r_2, r_3, ... are required in the elastoplastic problem.

We will seek the equations of the curves L_ϵ^+, L_ϵ in the form

$$y=\rho_\epsilon^+(x)=\epsilon^2\rho_2^+\left(\frac{x}{\epsilon}\right)+\epsilon^3\rho_3^+\left(\frac{x}{\epsilon}\right)+\dots \tag{4.3}$$

$$y=\rho_\epsilon(x)=\epsilon^2\rho_2(x)+\epsilon^3\rho_3(x)+\dots,$$

where $\rho_i^+(x/\epsilon)$ and $\rho_i(x)$ are unknown quantities.

Discontinuities in the derivatives of the stress field on the curve L_ϵ^-. The stress field is continuous on the characteristic L_ϵ^- but its derivatives, generally speaking, lose continuity. The magnitudes of the corresponding jumps are subsequently used to represent the stress field in the zone P_ϵ^+.

Let us give expressions for them in terms of the minimum number of independent quantities.

The relationships on the characteristic $y=f(x,\eta)$ of the system of Eqs (4.1), (1.1) have the form

$$\left[\frac{\partial\sigma_x}{\partial y}\right]=A,\ \left[\frac{\partial\sigma_{xy}}{\partial y}\right]=A\kappa,\ \left[\frac{\partial\sigma_y}{\partial y}\right]=A\kappa^2 \tag{4.4}$$

$$\kappa(x)=\frac{\partial f}{\partial x}(x,\eta)\left(1+\frac{f(x,\eta)}{R(x)}\right)^{-1}$$

We present the transport equation [7], which controls the values $A(x)$ in the variables ρ and ψ

$$\sigma_y + \sigma_x = 2\sigma, \ \sigma_y - \sigma_x = \sin 2\psi, \ 2\sigma_{xy} = \cos 2\psi \tag{4.5}$$

The characteristic L_ϵ^-, for which the transport equation is written, has a slope close to zero and belongs to the first or second of the families

$$\frac{dy}{dx} = (1 + \frac{y}{R(x)}) \operatorname{tg} \psi, \ \frac{dy}{dx} = -(1 + \frac{y}{R(x)}) \operatorname{ctg} \psi$$

depending on the value $\sigma_{xy}^p(0, 0) = 1/2$ or $\sigma_{xy}^p(0, 0) = -1/2$. To be specific, let us confine ourselves to the case when $\sigma_{xy}^p(0, 0) = -1/2$ and, consequently, $\omega(0, 0) = \pi/2$ (this is realized, in particular, in the example given in Sec. 1). Then, the transport equation

$$\frac{d}{dx}[\frac{\partial\psi}{\partial y}] = (-\frac{\partial c}{\partial y} + \frac{1}{2}\frac{\partial c}{\partial\psi}(\frac{\partial\sigma}{\partial y} - \frac{\partial\psi}{\partial y})_- - \frac{\partial c}{\partial\psi}(\frac{\partial\sigma}{\partial y} +$$

$$+ \frac{\partial\psi}{\partial y})_-)[\frac{\partial\psi}{\partial y}] - \frac{\partial c}{\partial\psi}[\frac{\partial\psi}{\partial y}]^2, \ c = -(1 + \frac{y}{R}) \operatorname{ctg} \psi$$

when account is taken of the relationship on the characteristic $[\partial\sigma/\delta y] = [\partial\psi/\partial y]$ and of the substitution (4.5), is transformed into the form

$$dA/dx = k_1(x)A + k_2(x)A^2, \ A = [\partial\sigma_x/\partial y]$$

Using its solution, the discontinuities in (4.4) are expressed in terms of the value of $A(0)$ at the point $x = 0$, $y = r$ of the characteristic L_ϵ^-.

Let us find the solution of this equation in the form of an expansion with respect to the x coordinate in the neighbourhood of the point $x = 0$. The coefficients of the expansion, as well as the expressions for $k_1(x)$ and $k_2(x)$ in the equation and the value of $A(0)$, depend on the quantity $r = r_2\epsilon^2 + \ldots$, and therefore, in their turn, we expand them with respect to the parameter ϵ. As a result of the calculations, we find that (by virtue of the estimate of the dimensions of the zone P_ϵ^+, the expansion coefficients are of the order of unity and terms of higher order of smallness than ϵ^2 are not written)

$$\begin{aligned} A(x) &= A_0 + \epsilon A_0' \frac{x}{\epsilon} + \epsilon^2 (A_1 r_2 + \frac{1}{2} A_0'' \frac{x^2}{\epsilon^2}) + \ldots \\ A_0' &= \left\{\frac{1}{2}\frac{\partial\sigma_y^p}{\partial y}(0,0) - \frac{\partial\sigma_x^p}{\partial y}(0,0)\right\}A_0 - \frac{1}{2}A_0^2 \\ A_0'' &= \left\{\frac{1}{2}\frac{\partial^2\sigma_y^p}{\partial x\partial y}(0,0) - \frac{\partial^2\sigma_x^p}{\partial x\partial y}(0,0)\right\}A_0 + \frac{1}{2}\left\{\frac{\partial\sigma_y^p}{\partial y}(0,0) - 2\frac{\partial\sigma_x^p}{\partial y}(0,0) - 2A_0\right\}A_0' \end{aligned} \tag{4.6}$$

The coefficients $A_0, A_1, \ldots$ of the expansion of the discontinuity $[\partial\sigma/\partial y]|_{x=0}$ are required in the elastoplastic problem.

For the quantity $\kappa(x)$ we use its expansion

$$\kappa(x) = \epsilon^2 (\frac{1}{2} f_{xxx} \frac{x^2}{\epsilon^2} + r_2 f_{x\eta}) + \ldots$$

According to formula (4.4) together with the representation (4.6), it leads to expressions for the discontinuities on the characteristic L_ϵ^- (terms of a higher order of smallness than ϵ^2 are neglected)

$$[\frac{\partial\sigma_x}{\partial y}] = A_0 + \epsilon A_0' \frac{x}{\epsilon} + \epsilon^2 (A_1 r_2 + \frac{1}{2} A_0'' \frac{x^2}{\epsilon^2}) + \dots$$

$$[\frac{\partial\sigma_{xy}}{\partial y}] = \epsilon^2 A_0 (\frac{1}{2} f_{xxx} \frac{x^2}{\epsilon^2} + r_2 f_{x\eta}) + \dots, \quad [\frac{\partial\sigma_y}{\partial y}] = 0 + \dots \tag{4.7}$$

The quantities

$$f_{x\eta} = -\frac{1}{2} \frac{\partial(\sigma_y^p - \sigma_x^p)}{\partial y}(0,0); \; f_{xxx} = -\frac{1}{2} \frac{\partial^2(\sigma_y^p - \sigma_x^p)}{\partial x^2}(0,0) \tag{4.8}$$

are found using the equation of the characteristic.

The discontinuities in the second-order derivatives (and, if required, the discontinuities in the higher derivatives) are expressed in terms of the parameters B_0, B_1, . . . while the quantities which are analogous to the quantities A_0, A_1, . . . are also expressed in terms of the A_i themselves. We note, in particular, that the expression which is subsequently used (terms that are small compared with unity are neglected) is

$$[\frac{\partial^2\sigma_{xy}}{\partial y^2}] = A_0 (-\frac{\partial(\sigma_y^p - \sigma_x^p)}{\partial y}(0,0) + \frac{1}{2} A_0) + \dots \tag{4.9}$$

Representation of the stress field. In the zone P_ϵ^+ we expand the stress field $s_\epsilon(x, y)$ with respect to the second argument $y = \rho_\epsilon^-(x) + (y - \rho_\epsilon^-(x))$ in the neighbourhood of its value $\rho_\epsilon^-(x)$. Using the continuity of the stresses on the curve L_ϵ^-, $\sigma_p(x, \rho_\epsilon^-(x)) = s_\epsilon(x, \rho_\epsilon^-(x))$ and the representation of the derivatives

$$\frac{\partial s_\epsilon}{\partial y}(x, \rho_\epsilon^-(x)) = \frac{\partial\sigma^p}{\partial y}(x, \rho_\epsilon^-(x)) + [\frac{\partial\sigma}{\partial y}](x)$$

where the discontinuities are taken on the contour L_ϵ^- and the analogous representation for the higher derivatives, we find (terms of a higher order of smallness than ϵ^4 are neglected)

$$s_\epsilon(x, y) = \sigma^p(x, 0) + \frac{\partial\sigma^p}{\partial y}(x, 0)y + \frac{1}{2}\frac{\partial^2\sigma^p}{\partial y^2}(x, 0)y^2 +$$

$$+ \frac{1}{2}[\frac{\partial\sigma}{\partial y}](x)(y - \rho_\epsilon^-(x)) + \frac{1}{2}[\frac{\partial^2\sigma}{\partial y^2}](x)(y - \rho_\epsilon^-(x))^2 + \dots \tag{4.10}$$

This representation together with expressions (4.7) for the discontinuities in the derivatives and the expressions which are analogous to those for the discontinuities of the higher derivatives reduce the problem of constructing the stress field s_ϵ to that of finding the quantities A_0, A_1, . . . , B_0, B_1, . . .

We shall see the stress field σ_ϵ^e in the elastic zone E_ϵ in the form

$$\sigma_\epsilon^e(x, y) = \sigma_0^e(x, y) + \epsilon^2\sigma_2^e(x, y) + \epsilon^3\sigma_3^e(x, y) + \epsilon^4\sigma_4^e(x, y) + \dots \tag{4.11}$$

where $\sigma_0^e(x, y) = \sigma_{\tau_0}(x, y)$ is the solution of the matching problem in the case of the critical load, and the fields σ_2^e and σ_3^e are of the order of magnitude of unity and, possibly, also depend on ϵ.

Matching conditions on the boundary of the elastic and plastic zones. On the contour $L_\epsilon \cup L_\epsilon^+$ which separates the elastic and the plastic zones, the stresses are continuous. On the curve L_ϵ, the continuity conditions $\sigma_\xi^e(x, \rho_\epsilon(x)) = \sigma^p(x, \rho_\epsilon(x))$ are written as in [8, 9] on the basis of the expansions (4.3), (4.11) and relationships following from (4.1) (the notation $\sigma_0 = \sigma_0^e - \sigma^p$ is used)

$$(\partial \sigma_{0xy}/\partial y)(x,0) = 0, \quad (\partial \sigma_{0y}/\partial y)(x,0) = 0$$

In the case of the first terms of expansion (4.1), the continuity of the stresses leads to the following equations (for values of x that correspond to a point of the curve L_ϵ)

$$\sigma_{2xy}^e(x,0) = 0, \sigma_{2y}^e(x,0) = 0, \frac{\partial \sigma_{0x}}{\partial y}(x,0)\rho_2(x) + \sigma_{2x}^e(x,0) = 0$$

$$\sigma_{3xy}^e(x,0) = 0, \sigma_{3y}^e(x,0) = 0, \frac{\partial \sigma_{0x}}{\partial y}(x,0)\rho_3(x) + \sigma_{3x}^e(x,0) = 0$$

$$\sigma_{4xy}^e(x,0) = -\frac{\partial \sigma_{2xy}^e}{\partial y}(x,0)\rho_2(x) - \frac{1}{2}\frac{\partial^2 \sigma_{0xy}}{\partial y^2}(x,0)\rho_2^2(x) \tag{4.12}$$

$$\sigma_{4y}^e(x,0) = -\frac{\partial \sigma_{2y}^e}{\partial y}(x,0)\rho_2(x) - \frac{1}{2}\frac{\partial^2 \sigma_{0y}}{\partial y^2}(x,0)\rho_2^2(x)$$

$$\frac{\partial \sigma_{0x}}{\partial y}(x,0)\rho_4(x) + \sigma_{4x}^e(x,0) + \frac{\partial \sigma_{2x}}{\partial y}(x,0)\rho_2(x) + \frac{1}{2}\frac{\partial^2 \sigma_{0x}}{\partial y^2}(x,0)\rho_2^2(x) = 0$$

The condition of continuity on the curve L_ϵ^+, $s_\xi(x, \rho_\epsilon^+(x)) = \sigma_\epsilon^e(x, \rho_\epsilon(x))$ is written on the basis of expansion (4.10), (4.11). Here, taking account of the estimate of the size of the zone P_ϵ^+, an additional expansion with respect to x is carried out in the neighbourhood of the point $x = 0$ and relationships (4.7) are also used. The continuity of the stresses leads to matching conditions which are identically satisfied for terms of the order of unity and have the form for terms of the order of ϵ^2 (at x values to which points on the curve L_ϵ^+ correspond)

$$\sigma_{2xy}^e(x,0) = 0, \sigma_{2y}^e(x,0) = 0$$

$$\frac{\partial \sigma_{0x}}{\partial y}(0,0)\rho_2^+\left(\frac{x}{\epsilon}\right) + \sigma_{2x}^e(0,0) - A_0\left(\rho_2^+\left(\frac{x}{\epsilon}\right) - \rho_2^-\left(\frac{x}{\epsilon}\right)\right) = 0 \tag{4.13}$$

For terms of the order of ϵ^3, the matching conditions have the form

$$\sigma_{3xy}^e(x,0) = 0, \sigma_{3y}^e(x,0) = 0$$

$$\frac{\partial \sigma_{0x}}{\partial y}(0,0)\rho_3^+\left(\frac{x}{\epsilon}\right) + \frac{\partial^2 \sigma_{0x}}{\partial x \partial y}(0,0)\frac{x}{\epsilon}\rho_2^+\left(\frac{x}{\epsilon}\right) + \frac{\partial \sigma_{2x}^e}{\partial x}(0,0)\frac{x}{\epsilon} + \tag{4.14}$$

$$+ \sigma_{3x}^e(0,0) - A_0\left(\rho_3^+\left(\frac{x}{\epsilon}\right) - \rho_3^-\left(\frac{x}{\epsilon}\right)\right) - A_0'\frac{x}{\epsilon}\left(\rho_2^+\left(\frac{x}{\epsilon}\right) - \rho_2^-\left(\frac{x}{\epsilon}\right)\right) = 0.$$

For terms of the order of ϵ^4, the matching conditions have the form

$$\sigma_{4xy}^e(x,0) = -\frac{1}{2}\frac{\partial^2 \sigma_{0xy}}{\partial y^2}(0,0)\left(\rho_2^+\left(\frac{x}{\epsilon}\right)\right)^2 - \frac{\partial \sigma_{2xy}^e}{\partial y}(0,0)\rho_2^+\left(\frac{x}{\epsilon}\right) +$$

$$+A_0\left(\frac{1}{2}f_{xxx}\frac{x^2}{\epsilon^2}+f_{x\eta}r_2\right)\left(\rho_2^+\left(\frac{x}{\epsilon}\right)-\rho_2^-\left(\frac{x}{\epsilon}\right)\right)+\frac{1}{2}\left[\frac{\partial^2\sigma_{xy}}{\partial y^2}\right]\left(\rho_2^+\left(\frac{x}{\epsilon}\right)-\rho_2^-\left(\frac{x}{\epsilon}\right)\right)^2$$

$$\sigma_{4y}^e(x,0)=-\frac{1}{2}\frac{\partial^2\sigma_{0y}}{\partial y^2}(0{,}0)\left(\rho_2^+\left(\frac{x}{\epsilon}\right)\right)^2-\frac{\partial\sigma_{2y}^e}{\partial y}(0{,}0)\,\rho_2^+\left(\frac{x}{\epsilon}\right)+ \tag{4.15}$$

$$+\frac{1}{2}\left[\frac{\partial^2\sigma_y}{\partial y^2}\right]\left(\rho_2^+\left(\frac{x}{\epsilon}\right)-\rho_2^-\left(\frac{x}{\epsilon}\right)\right)^2$$

$$\frac{\partial\sigma_{0x}}{\partial y}(0{,}0)\,\rho_4^+\left(\frac{x}{\epsilon}\right)+\frac{\partial^2\sigma_{0x}}{\partial x\partial y}(0{,}0)\frac{x}{\epsilon}\rho_3^+\left(\frac{x}{\epsilon}\right)+$$

$$+\frac{1}{2}\frac{\partial^3\sigma_{0x}}{\partial x^2\partial y}(0{,}0)\frac{x^2}{\epsilon^2}\rho_2^+\left(\frac{x}{\epsilon}\right)+\frac{1}{2}\frac{\partial^2\sigma_{0x}}{\partial y^2}(0{,}0)\left(\rho_2^+\left(\frac{x}{\epsilon}\right)\right)^2+$$

$$+\frac{1}{2}\frac{\partial^2\sigma_{2x}^e}{\partial x^2}(0{,}0)\frac{x^2}{\epsilon^2}+\frac{\partial\sigma_{2x}^e}{\partial y}(0{,}0)\,\rho_2^+\left(\frac{x}{\epsilon}\right)+\frac{\partial\sigma_{3x}^e}{\partial x}(0{,}0)\frac{x}{\epsilon}+$$

$$+\sigma_{4x}^e(x,0)-A_0\left(\rho_4^+\left(\frac{x}{\epsilon}\right)-\rho_4^-\left(\frac{x}{\epsilon}\right)\right)-A_0'\frac{x}{\epsilon}\left(\rho_3^+\left(\frac{x}{\epsilon}\right)-\rho_3^-\left(\frac{x}{\epsilon}\right)\right)-$$

$$-\left(A_1r_2+\frac{1}{2}A_0''\frac{x^2}{\epsilon^2}\right)\left(\rho_2^+\left(\frac{x}{\epsilon}\right)-\rho_2^-\left(\frac{x}{\epsilon}\right)\right)-\frac{1}{2}\left[\frac{\partial^2\sigma_x}{\partial y^2}\right]\left(\rho_2^+\left(\frac{x}{\epsilon}\right)-\rho_2^-\left(\frac{x}{\epsilon}\right)\right)^2=0$$

Here, in the expressions for the discontinuities in the second derivatives, it is understood that terms which are small compared with unity, such as, for example, in formula (4.9), are neglected.

Representation of the boundary conditions. The load at infinity (or on the outer boundary of the domain under consideration) is specified as a function of the parameter τ. Its expansion with respect to the small parameter ϵ^2 leads to the boundary conditions for the fields σ_2^e, σ_3^e, ... in the expansion (4.11).

For example, in the case of the biaxial stretching of a plane with a hole by forces p and q at infinity

$$\sigma_x\to p,\ \sigma_y\to q,\ \sigma_{xy}\to 0\ \text{ when }\ x^2+y^2\to\infty$$

and the corresponding boundary conditions for the first terms of expansion (4.11) have the form (when $x^2+y^2\to\infty$)

$$\begin{aligned}&\sigma_{2x}^e\to\frac{dp}{d\tau}(\tau_0),\ \sigma_{2y}^e\to\frac{dq}{d\tau}(\tau_0),\ \sigma_{2xy}^e\to 0\\&\sigma_{3x}^e\to 0,\ \sigma_{3y}^e\to 0,\ \sigma_{3xy}^e\to 0\\&\sigma_{4x}^e\to\frac{1}{2}\frac{d^2p}{d\tau^2}(\tau_0),\ \sigma_{4y}^e\to\frac{1}{2}\frac{d^2q}{d\tau^2}(\tau_0),\ \sigma_{4xy}^e\to 0\end{aligned} \tag{4.16}$$

Successive approximations. When there is no plastic zone P_ϵ^+ with a stress field different from σ^p the matching conditions for the components σ_{nxy}^e, σ_{ny}^e do not contain an unknown expansion coefficient ρ_n which describes the boundary of the plastic zone [8, 9] in the same approximation. Conditions (4.13)–(4.15) as well as the matching conditions in subsequent approximations also possess a similar property, and this also applies to the case under consideration. However, the procedure for constructing the subsequent approximation which is used when there is no zone P_ϵ^+ (the matching conditions for the components σ_{nxy}^e and σ_{ny}^e are used to find the field σ_n^e from the elastic problem and the matching condition for the component σ_{nx}^{ϵ} is then used to find the coefficient ρ_n) is additionally required in the case under consideration.

The matching conditions where there is a zone P_ϵ^+, generally speaking, contain parameters that describe the stress field in this zone and the positions of the boundary of the P_ϵ^- and P_ϵ^+ zones. These quantities, such as the parameter A_0 in condition (4.13), for example, may be incompletely determined by the preceding approximations. In order to find their values, it is necessary to use the admissibility conditions for the stresses (the fact that they do not pass beyond the yield surface) in the elastic zone. This condition guarantees the uniqueness of the solution of the plane elastoplastic problem [6] and therefore enables us to determine the values of all the free parameters.

5. THE STRESS FIELD IN THE ELASTIC ZONE AND THE BOUNDARY OF THE ZONE P_ϵ^-

The first unknown term in expansion (4.11) of the stress field in the elastic zone is defined by the solution $\sigma_2^3(x, y)$ of the elastic problem in the exterior of the contour $L_0 = l_{\tau_0}$, $y=0$ with the following boundary conditions. By virtue of relationships (4.12) and (4.13), a load-free condition must be satisfied over the whole of the contour L_0 and the first of the boundary conditions of the type (4.16) must also be satisfied.

The solution of this problem is

$$\sigma_2^e(x, y) = \partial\sigma_\tau(x, y)/\partial\tau|_{\tau=\tau_0} \tag{5.1}$$

where $\sigma_\tau(x, y)$ is the solution of the matching problem with the standard field σ^p for the value of the load parameter τ (satisfaction of the boundary conditions on the contour L_0 is checked using the geometrical conditions of compatibility on the matching contour l_τ which varies with the parameter τ).

The coefficient $\rho_2(x)$ in the representation of the curve L_ϵ is found from the matching condition (4.12) for the component σ_{2x}^e

$$\rho_2(x) = \rho_2^c(x) = -\sigma_{2x}^e(x, 0)(\partial\sigma_{0x}(x, 0)\partial y)^{-1} \tag{5.2}$$

where ρ_2^c is the first term of the expansion in the representation $y = \rho_\epsilon^c(x) = \epsilon^2\rho_2^0(x) \pm \ldots$ of the matching contour $l_{\tau_0+\epsilon^2}$ of the solution of the matching problem.

The quantities $\sigma_3^e(x, y)=0$ and $\rho_3(x)=0$ are found in a similar manner using conditions (4.12) and (4.14). Hence, the stress field in the elastic zone and the equation of the curve L_ϵ have the form

$$\begin{gathered}\sigma_\epsilon^e(x, y) = \sigma_0^e(x, y) + \epsilon^2\sigma_2^e(x, y) + O(\epsilon^4) \\ \sigma_2^e = \partial\sigma_\tau/\partial\tau|_{\tau=\tau_0},\ y = \rho_\epsilon(x) = \epsilon^2\rho_2(x) + O(\epsilon^4) \\ \rho_2(x) = \rho_2^c(x)\end{gathered} \tag{5.3}$$

The equation of the curve L_ϵ which has been found also enables one to find, to a first approximation, the equation of the remaining part of the boundary of the zone P_ϵ^-, that is, the remaining part of the characteristic L_ϵ^-. In fact, in accordance with the estimate (Sec. 4), the size of the zone P_ϵ^- in the x direction (the distance between common points of the curve L_ϵ and the characteristic L_ϵ^-) is of the order of ϵ. This is only possible when

$$r_2 = \rho_2^c(0) = -\sigma_{2x}^e(0,0)((\partial\sigma_{0x}/\partial y)(0,0))^{-1} \tag{5.4}$$

Next, we will find approximations for the boundary of the plastic zone, the stress field s_ϵ in the zone P_ϵ^+ and the characteristic L_ϵ^- which bounds it. In other words, let us find the function $\rho_2^+(x/\epsilon)$ and the parameter A_0 and also the function ρ_3^+ and the parameter r_3 that satisfy the matching conditions (4.13) and (4.14) for the component σ_x^ϵ. As was noted in Sec. 4, the matching conditions do not remove the arbitrariness in determination of these quantities and it is necessary to use the stress admissibility condition.

6. THE STRESS ADMISSIBILITY CONDITION. THE BOUNDARY OF THE PLASTIC ZONE

By virtue of relationship (5.3), the stress field $\sigma_\epsilon^e(x, y)$ is close to the solution of the matching problem, $\sigma_\epsilon^e(x, y)=\sigma_{\tau_0+\epsilon^2}(x, y)+O(\epsilon^4)$. A violation of the stress admissibility condition is therefore only possible in the neighbourhood of this domain where the solution of the matching problem $\sigma_{\tau_0+\epsilon^2}$ reaches beyond the yield surface, that is, close to the point $x=0$, $y=0$. On account of this, in using the stress admissibility condition, we consider an expansion with respect to x, y, and ϵ of the function

$$F(\sigma_\epsilon^e(x,y))=F(\sigma_0^e(x,y)+\epsilon^2\sigma_2^e(x,y)+\epsilon^4\sigma_4^e(x,y)+\dots)$$

in the neighbourhood of the point $x=0$, $y=0$.

Using the boundary conditions (4.14), (4.15) and the equality which follows from the results in Sec. 2 at the point $x=0$, $y=0$

$$\sigma_y^p-\sigma_x^p=0,\ \partial(\sigma_y^p-\sigma_x^p)/\partial x=0,\ \partial\sigma_{xy}^p/\partial x=0$$
$$\partial\sigma_{0xy}/\partial y=0,\ \partial^2\sigma_{0xy}^e/\partial x\partial y=0$$

we obtain the expression (terms of a higher order of smallness than ϵ^4 are neglected)

$$\begin{aligned}F(\sigma^e(x,y))&=ay^2+(b_1x^2+b_2\epsilon^2)y+c_1x^2\epsilon^2+\\&+c_{21}\Big\{d_1\big(\rho_2^+(\tfrac{x}{\epsilon})\big)^2+d_2\rho_2^+(\tfrac{x}{\epsilon})+A_0\big(\tfrac{1}{2}f_{xxx}\tfrac{x^2}{\epsilon^2}+f_{x\eta}r_2\big)\big(\rho_2^+(\tfrac{x}{\epsilon})-r_2\big)+\\&+\frac{1}{2}\Big[\frac{\partial^2\sigma_{xy}}{\partial y^2}\Big]\big(\rho_2^+(\tfrac{x}{\epsilon})-r_2\big)^2\Big\}\epsilon^4+c_{22}\epsilon^4+\dots\end{aligned}\tag{6.1}$$

$$a=4\sigma_{xy}^p\frac{\partial^2\sigma_{0xy}^e}{\partial y^2}+\Big(\frac{\partial\sigma_{0y}^e}{\partial y}-\frac{\partial\sigma_{0x}^e}{\partial y}\Big)^2$$

$$b_1=\frac{\partial^2(\sigma_y^p-\sigma_x^p)}{\partial x^2}\Big(\frac{\partial\sigma_{0y}^e}{\partial y}-\frac{\partial\sigma_{0x}^e}{\partial y}\Big)+4\sigma_{xy}^p\frac{\partial^3\sigma_{0xy}^e}{\partial x^2\partial y}$$

$$b_2=-2\sigma_{2x}^e\Big(\frac{\partial\sigma_{0y}^e}{\partial y}-\frac{\partial\sigma_{0x}^e}{\partial y}\Big)+8\sigma_{xy}^p\frac{\partial\sigma_{2xy}^e}{\partial y}$$

$$c_1=-\sigma_{2x}^e\frac{\partial^2(\sigma_y^p-\sigma_x^p)}{\partial x^2},\ c_{21}=8\sigma_{xy}^p,\ c_{22}=(\sigma_{2x}^e)^2$$

$$d_1=-\frac{1}{2}\frac{\partial^2\sigma_{0xy}}{\partial y^2},\ d_2=-\frac{\partial\sigma_{2xy}^e}{\partial y}$$

The stresses and their derivatives in the formulae for the coefficients a, b_1, b_2, c_1, c_{21}, c_{22}, d_1 and d_2 are calculated at the point $x=0$, $y=0$. Formulae (4.8) and (4.9) hold for the quantities $f_{x\eta}$, f_{xxx} and $[\partial^2\sigma_{xy}/\partial y^2]$ and the quantities $\rho_2^+(x/\epsilon)$ and A_0 are connected by relationship (4.13).

The stress admissibility condition. Let us consider equation $F(\sigma_\epsilon^e(x, y))=0$ in y for a fixed value of the coordinate x. The curves $y(x)$, that correspond to the roots of this equation define the domain in which the stresses appear beyond the yield surface $F(\sigma_\epsilon^\epsilon(x, y)) > 0$. By equating expression (6.1) to zero, it is possible to verify that $y_1 = \epsilon^2 \rho_2^+$ is one of the roots of this equation when $\sigma_{2x}^\epsilon(0, 0) \neq 0$ (this is natural, since the curve $y = \rho_\epsilon^+(x) = \epsilon^2 \rho_2^+(x/\epsilon) + \ldots$ is the boundary of the plastic zone on which the equality $F(\sigma_\epsilon^\epsilon(x, y)) = 0$ is satisfied by virtue of the continuity of the stresses). The second root of the equation being considered is then

$$y_2 = -\epsilon^2 (a^{-1}(b_1 x^2/\epsilon^2 + b_2) + \rho_2^+(x/\epsilon)) \tag{6.2}$$

The stress admissibility condition in the zone E_ϵ means that the domain in which $F(\sigma_\epsilon^\epsilon(x, y)) = 0$ does not intersect it. This is equivalent to the condition $y_2 \leqslant y_1$, that is, to the inequality

$$\rho_2^+(x/\epsilon) - a^{-1}(b_1 x^2/\epsilon^2 + b_2) \leqslant \rho_2^+(x/\epsilon) \tag{6.3}$$

The condition for the admissibility of the solution of the matching problem at a subcritical load and the estimation of the dimensions of the zone $V_{\tau_0+\epsilon^2}$ *for a supercritical load.* If the construction of Sec. 5 is carried out assuming that the zone P_ϵ^+ is not present and $\rho_\epsilon^+(x) = \rho_\epsilon(x)$, then it leads to the solution of the matching problem. For values of the load parameter $\tau_0 \pm \epsilon^2$, this solution is given by the expansion $\sigma_0^c(x, y) \pm \epsilon^2 \sigma_2^c(x, y) + \ldots$. Correspondingly, formula (6.1), if one puts $\rho_2^+(x/\epsilon) = r_2$ in it and, when $\tau = \tau_0 - \epsilon^2$, one replaces ϵ^2 by $-\epsilon^2$, yields the expansion of the function $F(\sigma_{\tau_0 \pm \epsilon^2}(x, y))$. One of the roots of the equation which determines the boundary of the domain in which the stresses $\sigma_{\tau_0 \pm \epsilon^2}$ appear beyond the yield surface is $y_1(x) = \pm \epsilon^2 r_2$ while the other, that is, $y_2(x)$ is found using formula (6.2) with ρ_2^+ replaced by r_2 (and ϵ^2 replaced by $-\epsilon^2$ when $\tau = \tau_0 - \epsilon^2$).

In the case of a subcritical load, the solution of the matching problem $\sigma_{\tau_0-\epsilon^2}$ does not pass beyond the yield surface in the domain which lies outside of the matching contour $l_{\tau_0-\epsilon^2}$. To a first approximation, the stress admissibility condition $y_2(x) \leqslant y_1(x)$ reduces to the inequality

$$b_1/a \geqslant 0, \quad 2\rho_2^c(0) + b_2/a \leqslant 0 \tag{6.4}$$

In the case of a subcritical load, the solution of the matching problem $\sigma_{\tau_0+\epsilon^2}$ extends beyond the yield surface in the domain $V_{\tau_0+\epsilon^2}$. It is defined to a first approximation by the condition $y_1(x) < y < y_2(x)$. It is seen from this that the size of the domain $V_{\tau_0+\epsilon^2}$ is of the order of ϵ in the x direction and of the order of ϵ^2 in the y direction. The magnitude of $F(\sigma_{\tau_0+\epsilon^2}(x, y))$ is positive in the domain $V_{\tau_0+\epsilon^2}$ and is of the order of ϵ^4.

The extension of the plastic zone and the magnitude of A_0. Henceforth it is assumed that a strict inequality (the remark in Sec. 2 refers to this assumption) is satisfied in the second of relationships (6.4). Then, in the case of a subcritical load, the plastic zone of the solution of the elastoplastic problem extends into the exterior of the matching contour of the solution of the matching problem (in the neighbourhood of the point $x = 0$, $y = 0$), $\rho_2^+(0) > \rho_2^c(0)$. In fact, for a value of the load parameter $\tau_0 + \epsilon^2$, the matching contour in the matching problem is described by the relationship $y = \rho_\epsilon^c(x) = \epsilon^2 \rho_2^c(x) + \ldots$ and the relationship $\rho_2^c(0) < \rho_2^+(0)$ holds according to the admissibility criteria (6.3), (6.4) with the strict inequality.

We will show that $A_0 = (\partial\sigma_{0x}/\partial y)$ $(0, 0)$. Actually, the matching condition (4.13), when account is taken of the equality $\rho_2^- = r_2$ and formula (5.4), is equivalent to the relationship

$$\left(\frac{\partial \sigma_{0x}}{\partial y}(0,0) - A_0\right)\left(\rho_2^+\left(\frac{x}{\epsilon}\right) - r_2\right) = 0$$

We will show that, if the second factor vanishes, the stress admissibility condition is not satisfied everywhere in the zone E_ϵ. In fact, the plastic zone P_ϵ^+ is located between the curves L_ϵ^- and L_ϵ^+. If the factor under consideration vanishes, then, by virtue of formulae (4.2) and (5.4), the distance from the curve L_ϵ^+ to curve L_ϵ^- (and hence to the matching contour) is of the order of ϵ^3. At the same time, the distance from the boundary of the domain $V_{\tau_0+\epsilon^2}$ to the matching contour is of the order of ϵ^2. Consequently, part of the domain $V_{\tau_0+\epsilon^2}$ does not overlap the zone P_ϵ^+ and is located in the zone E_ϵ. In the domain $V_{\tau_0+\epsilon^2}$, the stress field $\sigma_{\tau_0+\epsilon^2}$ extends beyond the yield surface. By virtue of relationships (5.3), the stress field σ_ϵ^e also extends beyond the yield surface in the domain $V_{\tau_0+\epsilon^2} \cap E_\epsilon$. Hence, the stresses σ_ϵ^e are inadmissible and, consequently, the equality $\rho_2^+(x/\epsilon) - r^2 = 0$ cannot be satisfied. This means that the above-mentioned formula holds for A_0. The admissibility condition must also be verified during calculations in this case also.

By virtue of the equality $A_0(\partial\sigma_{0x}/\partial y)$ (0, 0), the last of the matching conditions (4.13) turns out to be satisfied. At the same time, the boundary of the plastic zone (the function describing its position is $\rho_2^+(x/\epsilon)$) has still not been found.

The boundary of the plastic zone. The last of the matching conditions for σ_2^ϵ and σ_3^ϵ and relationship (4.14) for the component σ_{3x}^ϵ which remains unsatisfied serve for determining the curve L_ϵ^+. When account is taken of the equality $A_0 = (\partial\sigma_{0x}/\partial y)(0, 0)$ and expression (4.2) for the quantity ρ_3^-, it assumes the form

$$\frac{x}{\epsilon}\left\{-M\rho_2^+\left(\frac{x}{\epsilon}\right) + \frac{1}{6}A_0 f_{xxx}\frac{x^2}{\epsilon^2} + N\right\} + A_0 r_3 = 0$$

$$M \equiv A_0' - \frac{\partial^2\sigma_{0x}}{\partial x\partial y}(0,0),\quad N \equiv \frac{\partial\sigma_{2x}^e}{\partial x}(0,0) + A_0 f_{x\eta} r_2 + A_0' r_2$$

(the magnitude of A_0' is defined by formula (4.6)). Whence, when $A_0 \neq 0$ and $M \neq 0$ (the remark in Sec. 2 refers to these conditions), we find

$$\rho_2^+\left(\frac{x}{\epsilon}\right) = \left(\frac{1}{6}A_0 f_{xxx}\frac{x^2}{\epsilon^2} + N\right)/M,\quad r_3 = 0 \tag{6.5}$$

Remark. The boundary of the zone P_ϵ^+ coincides neither with the matching contour $L_{\tau_0+\epsilon^2}$ in the matching problem nor with the boundary of the domain in which the solution of the matching problem passes beyond the yield surface.

Hence, in addition to the approximations for the stress field in the elastic zone and the segment of the boundary L_ϵ between the elastic and plastic zones (relationship (5.3)), corresponding approximations are found for: (1) the remaining part L_ϵ^+ of this boundary (relationship (6.5)), (2) the boundary between the plastic zones P_ϵ^- and P_ϵ^+, that is the curve L_ϵ^- (relationship (4.2) in which quantities determined from formulae (4.8), (5.4) and (6.5), occur), (3) the stress field in the zone P_ϵ^+. The latter is given by formula (4.10) in which, when terms of higher order of smallness than ϵ^3 are neglected, the remaining terms are completely determined by the quantities A_0, r_2 and r_3 which have been found, since they enable one to find the discontinuities (4.7) occurring in (4.10) and the function (4.2) $\rho_\epsilon^-(x)$ with the required accuracy.

7. DEVELOPMENT OF THE PLASTIC ZONE AROUND A CIRCULAR HOLE

Consider the biaxial loading at infinity $p = \tau p_0$, $q = \tau q_0$, which satisfies conditions (1.2) in the case of a plane with an unloaded circular hole (the example from Sec. 1). The value $\tau_0 = (\sqrt{(2)}-1)|q_0 - p_0|$ is the critical one. Using the known solution [1], it is possible to verify that there is no degeneracy (see the remark in Sec. 2) in this case and to calculate the value of (4.8), $A_0 = (\partial\sigma_{0x}/\partial y)(0, 0)$ and the other quantities occurring in formula (6.5).

The results of calculations of the boundaries of the plastic zone are shown in Fig. 5 as an example. This figure shows graphs of the functions $y = \epsilon^2\rho_2(x)$ (corresponding to the curve L_ϵ) and $y = \epsilon^2\rho_2^-(x/\epsilon) + \epsilon^3\rho_3^-(x/\epsilon)$ (corresponding to the curve L_ϵ^-) and $y = \epsilon^2\rho_2^+(x/\epsilon)$ (corresponding to the curve L_ϵ^+) as well as a

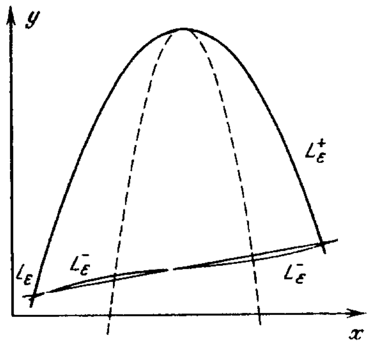

FIG. 5.

plot of the function $y=\epsilon^2\rho_2^-(x/\epsilon)$, represented by the dashed line, where $\rho_2^-(x/\epsilon)$ is the expression on the left-hand side of inequality (6.3). The calculations were carried out for $p_0=1.5$, $q_0=\sqrt{(2)}+0.5$ and $\epsilon^2=10^{-3}$. The arrangement of the curves shows that the stress admissibility condition is satisfied in the zone E_ϵ in the case of the problem being considered.

The curve L_ϵ, to a first approximation, is identical with an ellipse, representing the matching contour in the solution of the matching problem when $\tau=\tau_0+\epsilon^2$. The complete plastic zone in the solution of the elastoplastic problem is obtained to a first approximation by adding the part of the plane, the outer boundary of which is the curve $y=\epsilon^2\rho_2^+(x/\epsilon)$, to the "plastic zone" corresponding to the solution of the matching problem.

We note that, regardless of the values of p_0 and q_0, the development of the plastic zone P_ϵ^+ starts from a point which lies on the matching contour l_{τ_0} which corresponds to the critical load and lies on a ray which makes an angle of $\pi/8$ with the direction of the x axis. The development of similar zones also starts from points which are symmetrical about the above-mentioned zone with respect to the x and y axes.

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